PERGAMON

International Journal of Solids and Structures 38 (2001) 4921–4936

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Three-dimensional static analysis of multi-layered piezoelectric hollow spheres via the state space method

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Received 14 February 2000; in revised form 30 July 2000

Abstract

A general non-axisymmetric exact analysis of the statics of a laminated piezoelectric hollow sphere is presented in the paper by using a state space method. To select a proper set of state variables, three displacement functions and two stress functions are introduced. It is found that the basic equations of a spherically isotropic piezoelectric medium are eventually turned to two separated state equations with constant coefficients, the solutions of which are then obtained by virtue of matrix theory. The continuity conditions at each interface are then used to derive two relationships between respective boundary variables at the inner and outer spherical surfaces. No matter how many layers the sphere contains, the orders of the final solving equations remain unaltered. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Non-axisymmetric static behavior; Multi-layered piezoelectric hollow sphere; State space method; Cayley–Hamilton theorem

1. Introduction

Research on piezoelectric materials has been of great interest because of their wide applications in various industries. Numerous papers have been published on the statics and dynamics of plates and shells made of piezoelectric materials, see Dokmeci (1980), Tzou and Zhong (1994), Heyliger (1994, 1997), Chen et al. (1996), Bisegna and Maceri (1996), Paul and Natarajan (1996), and Ding et al. (1997), to name a few. In particular, as to spherical shells, Kirichok (1980) studied the radial oscillation of a piezoelectric spherical shell coupled with fluid media. Shul'ga (1993) investigated the general non-axisymmetric vibration of a piezoceramic hollow sphere by using a separation technique. Chen and Ding (1998) employed a displacement separation method to exactly analyze a rotating spherically isotropic piezoelectric spherical shell. Recently, Heyliger and Wu (1999) analyzed layered piezoelectric spheres, where analytical solution for the purely radial problem could be found.

Analysis based on state space formulations (also known as the method of initial functions) has been shown very powerful to deal with problems related to multi-layered structures (Lure, 1964; Das and Setlur,

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PII: S0020-7683(00)00314-0

1970; Kameswara Rao and Das, 1977; Ye and Soldatos, 1994). Sosa (1992) and Sosa and Castro (1993) first presented the two-dimensional state space formulations for plane problems of piezoelectric layered structures. Lee and Jiang (1996) and Chen et al. (1997) independently derived the three-dimensional static state space formula and analyzed the bending of piezoelectric plates. Chen et al. (1998) later published the dynamic space formula and considered the free vibration of a piezoelectric thick plate. The above mentioned state space formulae are all established in Cartesian coordinates. Ding et al. (1999) derived the state space equation in circular cylindrical coordinates and, in connection with the finite Hankel transform technique, considered the axisymmetric free vibration of a circular plate. Zhou et al. (1999) recently generalized the method proposed by Ye and Soldatos (1994) to analyze laminated piezoelectric cylindrical shells. There is no published works even related to the analysis of multi-layered elastic spherical shells/hollow spheres that is based on a state space method. Although Shul'ga et al. (1988) ever obtained two independent state space equations with varying coefficients in spherical coordinates for the vibration of a non-homogeneous spherically isotropic elastic hollow sphere, they just treated them as the simplified equations that were finally solved using a numerical procedure.

This paper intends to present an exact analysis of the statics of a multi-layered, spherically isotropic, piezoelectric hollow sphere by using a state space method. In fact, two independent state equations with constant coefficients are constructed after using a series of techniques. In particular, separation formulae for displacements and shear stresses are employed to simplify the basic equations. Relations between the state variables at the top and lower surfaces of each layer are established directly from the solutions of the state equations. Allowing for the continuity conditions between two adjacent layers, one can readily derive two relationships between the state variables at the inner and outer spherical boundary surfaces. The final solving equations are presented for a specified boundary value problem. It is shown that no matter how many layers there are, the orders of the solving equations are always the same. Numerical results are finally given to show the validity and effectiveness of the present method.

2. Basic equations

The basic equations for linear piezoelasticity for spherical isotropy can be found in Tiersten (1969), Shul'ga (1993), Chen and Ding (1998) and Chen (1999). For the sake of the followed analysis, these equations are rewritten in a different way here. In spherical coordinates (r, θ, ϕ), assuming the center of the spherical isotropy coincident with the origin, one can write out the constitutive relations as follows:

$$
\begin{aligned}
&\Sigma_{\theta\theta} = r\sigma_{\theta\theta} = c_{11}S_{\theta\theta} + c_{12}S_{\phi\phi} + c_{13}S_{rr} + e_{31}\nabla_2\Phi,\\
&\Sigma_{\phi\phi} = r\sigma_{\phi\phi} = c_{12}S_{\theta\theta} + c_{11}S_{\phi\phi} + c_{13}S_{rr} + e_{31}\nabla_2\Phi,\\
&\Sigma_{rr} = r\sigma_{rr} = c_{13}S_{\theta\theta} + c_{13}S_{\phi\phi} + c_{33}S_{rr} + e_{33}\nabla_2\Phi,\\
&\Sigma_{r\theta} = r\sigma_{r\theta} = 2c_{44}S_{r\theta} + e_{15}\frac{\partial\Phi}{\partial\theta},\\
&\Sigma_{r\phi} = r\sigma_{r\phi} = 2c_{44}S_{r\phi} + \frac{e_{15}}{\sin\theta}\frac{\partial\Phi}{\partial\phi},\\
&\Sigma_{\theta\phi} = r\sigma_{\theta\phi} = 2c_{66}S_{\theta\phi},\\
&\Delta_\theta = rD_\theta = 2e_{15}S_{r\theta} - \varepsilon_{11}\frac{\partial\Phi}{\partial\theta},\\
&\Delta_\phi = rD_\phi = 2e_{15}S_{r\phi} - \frac{\varepsilon_{11}}{\sin\theta}\frac{\partial\Phi}{\partial\phi},\\
&\Delta_r = rD_r = e_{31}S_{\theta\theta} + e_{31}S_{\phi\phi} + e_{33}S_{rr} - \varepsilon_{33}\nabla_2\Phi,
\end{aligned}
\tag{1}
$$

where $\nabla_2 = r\partial/\partial r$, σ_{ij} are the stress components, Φ and D_i are the electric potential and electric displacement components, respectively, c_{ij} are the elastic stiffness constants (measured in a constant electric field), ε_{ij} the dielectric constants (measured at constant strain), and e_{ij} the piezoelectric constants. It is noted here that an additional relationship $c_{11} = c_{12} + 2c_{66}$ holds for the spherical isotropy. S_{ij} in Eq. (1) are determined by

$$\begin{aligned}
S_{rr} &= rs_{rr} = \nabla_2 u_r, \\
S_{\theta\theta} &= rs_{\theta\theta} = \frac{\partial u_\theta}{\partial \theta} + u_r, \\
S_{\phi\phi} &= rs_{\phi\phi} = \frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial \phi} + u_r + u_\theta \cot\theta, \\
2S_{r\theta} &= 2rs_{r\theta} = \frac{\partial u_r}{\partial \theta} + \nabla_2 u_\theta - u_\theta, \\
2S_{r\phi} &= 2rs_{r\phi} = \frac{1}{\sin\theta}\frac{\partial u_r}{\partial \phi} + \nabla_2 u_\phi - u_\phi, \\
2S_{\theta\phi} &= 2rs_{\theta\phi} = \frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{\partial u_\phi}{\partial \theta} - u_\phi \cot\theta,
\end{aligned} \tag{2}$$

where s_{ij} are the strain components, u_i $(i = r, \theta, \phi)$ are components of the mechanical displacement. The equations of equilibrium are also rewritten in the following form:

$$\begin{aligned}
&\nabla_2 \Sigma_{r\theta} + \csc\theta \frac{\partial \Sigma_{\theta\phi}}{\partial \phi} + \frac{\partial \Sigma_{\theta\theta}}{\partial \theta} + 2\Sigma_{r\theta} + (\Sigma_{\theta\theta} - \Sigma_{\phi\phi})\cot\theta = 0, \\
&\nabla_2 \Sigma_{r\phi} + \csc\theta \frac{\partial \Sigma_{\phi\phi}}{\partial \phi} + \frac{\partial \Sigma_{\theta\phi}}{\partial \theta} + 2\Sigma_{r\phi} + 2\Sigma_{\theta\phi}\cot\theta = 0, \\
&\nabla_2 \Sigma_{rr} + \csc\theta \frac{\partial \Sigma_{r\phi}}{\partial \phi} + \frac{\partial \Sigma_{r\theta}}{\partial \theta} + \Sigma_{rr} - \Sigma_{\theta\theta} - \Sigma_{\phi\phi} + \Sigma_{r\theta}\cot\theta = 0.
\end{aligned} \tag{3}$$

In the absence of free charge density, the charge equation of electrostatics is

$$\nabla_2 \Delta_r + \Delta_r + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\Delta_\theta \sin\theta) + \frac{1}{\sin\theta}\frac{\partial \Delta_\phi}{\partial \phi} = 0. \tag{4}$$

3. The separation technique

To establish the state equation, we should first determine the state variables. An intuitive selection will be the three components of the mechanical displacement (u_r, u_θ, u_ϕ), the three components of the stress tensor $(\sigma_{rr}, \sigma_{r\theta}, \sigma_{r\phi})$, the electric potential Φ, and the radial component of the electric displacement D_r. However, such a selection will lead to additional confusion in the solving procedure. We therefore employ three displacement functions w, G and ψ to rewrite the mechanical displacements as follows (Chen, 1999):

$$u_\theta = -\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \phi} - \frac{\partial G}{\partial \theta}, \qquad u_\phi = \frac{\partial \psi}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial G}{\partial \phi}, \qquad u_r = w. \tag{5}$$

In connection with Eq. (5), we further assume (Shul'ga, 1993)

$$\Sigma_{r\theta} = -\frac{1}{\sin\theta}\frac{\partial \Sigma_1}{\partial \phi} - \frac{\partial \Sigma_2}{\partial \theta}, \qquad \Sigma_{r\phi} = \frac{\partial \Sigma_1}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial \Sigma_2}{\partial \phi}, \tag{6}$$

where Σ_1 and Σ_2 are two stress functions.

By employing Eqs. (5) and (6), we can transfer the first two equations in Eq. (3) into the following two equations:

$$\frac{\partial}{\partial\theta}\left[\nabla_2\Sigma_2+2\Sigma_2+c_{11}\nabla_1^2G+2c_{66}G-(c_{11}+c_{12})w-c_{13}\nabla_2w-e_{31}\nabla_2\Phi\right]+\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left[\nabla_2\Sigma_1+2\Sigma_1+c_{66}(\nabla_1^2\psi+2\psi)\right]=0, \tag{7}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left[\nabla_2\Sigma_2+2\Sigma_2+c_{11}\nabla_1^2G+2c_{66}G-(c_{11}+c_{12})w-c_{13}\nabla_2w-e_{31}\nabla_2\Phi\right]-\frac{\partial}{\partial\theta}\left[\nabla_2\Sigma_1+2\Sigma_1+c_{66}(\nabla_1^2\psi+2\psi)\right]=0, \tag{8}$$

where

$$\nabla_1^2=\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\csc^2\theta\frac{\partial^2}{\partial\phi^2}.$$

From Eqs. (7) and (8), one obtains without the loss of generality (Hu, 1954; Chen, 1996)

$$\nabla_2\Sigma_2+2\Sigma_2+c_{11}\nabla_1^2G+2c_{66}G-(c_{11}+c_{12})w-c_{13}\nabla_2w-e_{31}\nabla_2\Phi=0, \tag{9}$$

$$\nabla_2\Sigma_1+2\Sigma_1+c_{66}(\nabla_1^2\psi+2\psi)=0. \tag{10}$$

The third equation in Eq. (3) and Eq. (4) become

$$\nabla_2\Sigma_{rr}+\Sigma_{rr}-\nabla_1^2\Sigma_2+(c_{11}+c_{12})\nabla_1^2G-2(c_{11}+c_{12})w-2c_{13}\nabla_2w-2e_{31}\nabla_2\Phi=0, \tag{11}$$

$$\nabla_2\Delta_r+\Delta_r+e_{15}\nabla_1^2w-\varepsilon_{11}\nabla_1^2\Phi+e_{15}\nabla_1^2G-e_{15}\nabla_2\nabla_1^2G=0. \tag{12}$$

From the fourth and fifth equations in Eq. (1), by utilizing Eq. (2), one obtains

$$\frac{\partial}{\partial\theta}(\Sigma_2+c_{44}w+e_{15}\Phi-c_{44}\nabla_2G+c_{44}G)-\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(c_{44}\nabla_2\psi-c_{44}\psi-\Sigma_1)=0, \tag{13}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\Sigma_2+c_{44}w+e_{15}\Phi-c_{44}\nabla_2G+c_{44}G)+\frac{\partial}{\partial\theta}(c_{44}\nabla_2\psi-c_{44}\psi-\Sigma_1)=0. \tag{14}$$

From the above two equations, one can obtain similarly

$$\Sigma_2+c_{44}w+e_{15}\Phi-c_{44}\nabla_2G+c_{44}G=0, \tag{15}$$

$$c_{44}\nabla_2\psi-c_{44}\psi-\Sigma_1=0. \tag{16}$$

From the third and ninth equations in Eq. (1), by utilizing Eq. (2), one obtains

$$\Sigma_{rr}=-c_{13}\nabla_1^2G+2c_{13}w+c_{33}\nabla_2w+e_{33}\nabla_2\Phi, \tag{17}$$

$$\Delta_r=-e_{31}\nabla_1^2G+2e_{31}w+e_{33}\nabla_2w-\varepsilon_{33}\nabla_2\Phi. \tag{18}$$

4. Mathematical formulations of the state space method

Obviously, ψ and Σ_1 are uncoupled from the other state variables as one can see from Eqs. (10) and (16), which can be combined in the following way:

$$\nabla_2\begin{Bmatrix}\Sigma_1\\ \psi\end{Bmatrix}=\begin{bmatrix}-2 & -c_{66}(\nabla_1^2+2)\\ \frac{1}{c_{44}} & 1\end{bmatrix}\begin{Bmatrix}\Sigma_1\\ \psi\end{Bmatrix}. \tag{19}$$

From Eqs. (9), (11), (12), (15), (17) and (18), we obtain

$$\nabla_2\begin{Bmatrix}\Sigma_{rr}\\ \Sigma_2\\ G\\ w\\ \Delta_r\\ \Phi\end{Bmatrix}=\begin{bmatrix}2\beta-1 & \nabla_1^2 & k_1\nabla_1^2 & -2k_1 & 2\gamma & 0\\ \beta & -2 & k_2\nabla_1^2-2c_{66} & -k_1 & \gamma & 0\\ 0 & 1/c_{44} & 1 & 1 & 0 & e_{15}/c_{44}\\ \varepsilon_{33}/\alpha & 0 & \beta\nabla_1^2 & -2\beta & e_{33}/\alpha & 0\\ 0 & (e_{15}/c_{44})\nabla_1^2 & 0 & 0 & -1 & k_3\nabla_1^2\\ e_{33}/\alpha & 0 & \gamma\nabla_1^2 & -2\gamma & -c_{33}/\alpha & 0\end{bmatrix}\begin{Bmatrix}\Sigma_{rr}\\ \Sigma_2\\ G\\ w\\ \Delta_r\\ \Phi\end{Bmatrix}, \tag{20}$$

where

$$\alpha=c_{33}\varepsilon_{33}+e_{33}^2,\quad \beta=(c_{13}\varepsilon_{33}+e_{31}e_{33})/\alpha,\quad \gamma=(c_{13}e_{33}-c_{33}e_{31})/\alpha,$$
$$k_1=2(c_{13}\beta+e_{31}\gamma)-(c_{11}+c_{12}),\quad k_2=k_1/2-c_{66},\quad k_3=\varepsilon_{11}+e_{15}^2/c_{44}.$$

For a p-ply hollow sphere as shown in Fig. 1, the following variable substitution is taken for the ith spherical layer:

$$r=a_i\mathrm{e}^{\xi}\quad (i=1,2,\ldots,p;\ 0\leqslant\xi\leqslant\xi_i), \tag{21}$$

where $\xi_i=\ln(b_i/a_i)$, a_i and b_i are the inner and outer radii of the ith spherical layer, respectively. Making use of Eq. (21), Eqs. (19) and (20) become

$$\frac{\partial}{\partial\xi}\begin{Bmatrix}\Sigma_1\\ \psi\end{Bmatrix}=\begin{bmatrix}-2 & -c_{66}(\nabla_1^2+2)\\ \frac{1}{c_{44}} & 1\end{bmatrix}\begin{Bmatrix}\Sigma_1\\ \psi\end{Bmatrix}, \tag{22}$$

$$\frac{\partial}{\partial\xi}\begin{Bmatrix}\Sigma_{rr}\\ \Sigma_2\\ G\\ w\\ \Delta_r\\ \Phi\end{Bmatrix}=\begin{bmatrix}2\beta-1 & \nabla_1^2 & k_1\nabla_1^2 & -2k_1 & 2\gamma & 0\\ \beta & -2 & k_2\nabla_1^2-2c_{66} & -k_1 & \gamma & 0\\ 0 & 1/c_{44} & 1 & 1 & 0 & e_{15}/c_{44}\\ \varepsilon_{33}/\alpha & 0 & \beta\nabla_1^2 & -2\beta & e_{33}/\alpha & 0\\ 0 & (e_{15}/c_{44})\nabla_1^2 & 0 & 0 & -1 & k_3\nabla_1^2\\ e_{33}/\alpha & 0 & \gamma\nabla_1^2 & -2\gamma & -c_{33}/\alpha & 0\end{bmatrix}\begin{Bmatrix}\Sigma_{rr}\\ \Sigma_2\\ G\\ w\\ \Delta_r\\ \Phi\end{Bmatrix}. \tag{23}$$

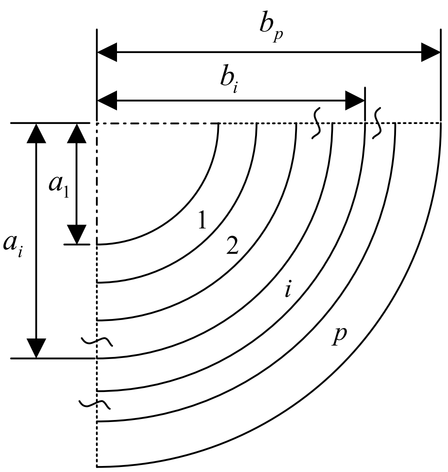

Fig. 1. The geometry of a p-ply hollow sphere.

The above two equations still include the partial differential operator ∇_1^2, which can be eliminated from Eqs. (22) and (23) by assuming

$$\begin{cases} \Sigma_1 = a_1 c_{44}^{(1)} \sum_{m=0}^{n}\sum_{n=1}^{\infty} T_{1n}(\xi) S_n^m(\theta,\phi), \qquad \psi = a_1 \sum_{m=0}^{n}\sum_{n=1}^{\infty} T_{2n}(\xi) S_n^m(\theta,\phi), \\ \Sigma_{rr} = a_1 c_{44}^{(1)} \sum_{m=0}^{n}\sum_{n=0}^{\infty} T_{3n}(\xi) S_n^m(\theta,\phi), \qquad \Sigma_2 = a_1 c_{44}^{(1)} \sum_{m=0}^{n}\sum_{n=0}^{\infty} T_{4n}(\xi) S_n^m(\theta,\phi), \\ G = a_1 \sum_{m=0}^{n}\sum_{n=0}^{\infty} T_{5n}(\xi) S_n^m(\theta,\phi), \qquad w = a_1 \sum_{m=0}^{n}\sum_{n=0}^{\infty} T_{6n}(\xi) S_n^m(\theta,\phi), \\ \Delta_r = a_1 e_{33}^{(1)} \sum_{m=0}^{n}\sum_{n=0}^{\infty} T_{7n}(\xi) S_n^m(\theta,\phi), \qquad \Phi = a_1 \frac{e_{33}^{(1)}}{\varepsilon_{33}^{(1)}} \sum_{m=0}^{n}\sum_{n=0}^{\infty} T_{8n}(\xi) S_n^m(\theta,\phi), \end{cases} \tag{24}$$

where $S_n^m(\theta,\phi) = P_n^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi}$ are the spherical harmonic functions and $P_n^m(x)$ are the associated Legendre polynomials, n and m are integers, $c_{44}^{(1)}$, $\varepsilon_{33}^{(1)}$ and $e_{33}^{(1)}$ represent the material constants in the first layer. Substituting Eq. (24) into Eqs. (22) and (23) gives

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{T}_{1ni} = \mathbf{M}_{1ni}\mathbf{T}_{1ni} \quad (n = 1,2,3,\ldots;\ i = 1,2,\ldots,p), \tag{25}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{T}_{2ni} = \mathbf{M}_{2ni}\mathbf{T}_{2ni} \quad (n = 0,1,2,\ldots;\ i = 1,2,\ldots,p), \tag{26}$$

where $\mathbf{T}_{1ni} = [T_{1n}, T_{2n}]_i^{\mathrm{T}} = [T_{1ni}, T_{2ni}]^{\mathrm{T}}$, $\mathbf{T}_{2ni} = [T_{3n}, T_{4n}, \ldots, T_{8n}]_i^{\mathrm{T}} = [T_{3ni}, T_{4ni}, \ldots, T_{8ni}]^{\mathrm{T}}$, and

$$\mathbf{M}_{1ni} = \begin{bmatrix} -2 & \frac{(l-2)c_{66}}{c_{44}^{(1)}} \\ \frac{c_{44}^{(1)}}{c_{44}} & 1 \end{bmatrix},$$

$$\mathbf{M}_{2ni} = \begin{bmatrix} 2\beta-1 & -l & -\frac{k_1 l}{c_{44}^{(1)}} & -\frac{2k_1}{c_{44}^{(1)}} & \frac{2\gamma e_{33}^{(1)}}{c_{44}^{(1)}} & 0 \\ \beta & -2 & -\frac{k_2 l + 2c_{66}}{c_{44}^{(1)}} & -\frac{k_1}{c_{44}^{(1)}} & \frac{\gamma e_{33}^{(1)}}{c_{44}^{(1)}} & 0 \\ 0 & \frac{c_{44}^{(1)}}{c_{44}} & 1 & 1 & 0 & \frac{e_{15} e_{33}^{(1)}}{c_{44}\varepsilon_{33}^{(1)}} \\ \frac{c_{44}^{(1)}\varepsilon_{33}}{\alpha} & 0 & -\beta l & -2\beta & \frac{e_{33} e_{33}^{(1)}}{\alpha} & 0 \\ 0 & -\frac{e_{15} c_{44}^{(1)} l}{e_{33}^{(1)} c_{44}} & 0 & 0 & -1 & -\frac{k_3 l}{\varepsilon_{33}^{(1)}} \\ \frac{e_{33} c_{44}^{(1)} \varepsilon_{33}^{(1)}}{\alpha e_{33}^{(1)}} & 0 & -\frac{\gamma \varepsilon_{33}^{(1)} l}{e_{33}^{(1)}} & -\frac{2\gamma\varepsilon_{33}^{(1)}}{e_{33}^{(1)}} & -\frac{c_{33}\varepsilon_{33}^{(1)}}{\alpha} & 0 \end{bmatrix}.$$

Here, the notation $l = n(n+1)$ is employed. At this stage, we have established in each layer two independent state equations with constant coefficients as shown in Eqs. (25) and (26).

5. The solutions

Solutions to Eqs. (25) and (26) can be obtained by using the matrix theory (Bellman, 1970) as follows:

$$\mathbf{T}_{1ni}(\xi) = \exp(\mathbf{M}_{1ni}\xi)\mathbf{T}_{1ni}(0) \quad (n = 1,2,3,\ldots;\ 0 \leqslant \xi \leqslant \xi_i), \tag{27}$$

$$\mathbf{T}_{2ni}(\xi) = \exp(\mathbf{M}_{2ni}\xi)\mathbf{T}_{2ni}(0) \quad (n = 0,1,2,\ldots;\ 0 \leqslant \xi \leqslant \xi_i), \tag{28}$$

where the exponential function matrices $\exp(\mathbf{M}_{1ni}\xi)$ and $\exp(\mathbf{M}_{2ni}\xi)$ are known as the transfer matrices. By virtue of the Cayley–Hamilton theorem (Bellman, 1970), one obtains

$$\exp(\mathbf{M}_{1ni}\xi) = \alpha_{0i}(\xi)\mathbf{I}_{2\times 2} + \alpha_{1i}(\xi)\mathbf{M}_{1ni}, \tag{29}$$

$$\exp(\mathbf{M}_{2ni}\xi) = \beta_{0i}(\xi)\mathbf{I}_{6\times 6} + \sum_{k=1}^{5}\beta_{ki}(\xi)\mathbf{M}_{2ni}^{k}, \tag{30}$$

where $\mathbf{I}_{2\times 2}$ and $\mathbf{I}_{6\times 6}$ are unit matrices of the second-order and sixth-order, respectively, and

$$\begin{Bmatrix} \alpha_{0i}(\xi) \\ \alpha_{1i}(\xi) \end{Bmatrix} = \begin{bmatrix} 1 & \lambda_{1i} \\ 1 & \lambda_{2i} \end{bmatrix}^{-1} \begin{Bmatrix} e^{\lambda_{1i}\xi} \\ e^{\lambda_{2i}\xi} \end{Bmatrix}, \tag{31}$$

$$\begin{Bmatrix} \beta_{0i}(\xi) \\ \beta_{1i}(\xi) \\ \beta_{2i}(\xi) \\ \beta_{3i}(\xi) \\ \beta_{4i}(\xi) \\ \beta_{5i}(\xi) \end{Bmatrix} = \begin{bmatrix} 1 & \eta_{1i} & \eta_{1i}^2 & \eta_{1i}^3 & \eta_{1i}^4 & \eta_{1i}^5 \\ 1 & \eta_{2i} & \eta_{2i}^2 & \eta_{2i}^3 & \eta_{2i}^4 & \eta_{2i}^5 \\ 1 & \eta_{3i} & \eta_{3i}^2 & \eta_{3i}^3 & \eta_{3i}^4 & \eta_{3i}^5 \\ 1 & \eta_{4i} & \eta_{4i}^2 & \eta_{4i}^3 & \eta_{4i}^4 & \eta_{4i}^5 \\ 1 & \eta_{5i} & \eta_{5i}^2 & \eta_{5i}^3 & \eta_{5i}^4 & \eta_{5i}^5 \\ 1 & \eta_{6i} & \eta_{6i}^2 & \eta_{6i}^3 & \eta_{6i}^4 & \eta_{6i}^5 \end{bmatrix}^{-1} \begin{Bmatrix} e^{\eta_{1i}\xi} \\ e^{\eta_{2i}\xi} \\ e^{\eta_{3i}\xi} \\ e^{\eta_{4i}\xi} \\ e^{\eta_{5i}\xi} \\ e^{\eta_{6i}\xi} \end{Bmatrix}, \tag{32}$$

here, λ_{ki} and η_{ki} are the eigenvalues of the two matrices $\mathbf{M}_{1ni}$ and $\mathbf{M}_{2ni}$, respectively. It is noted here that Eq. (31) or (32) is valid only when the eigenvalues λ_{ki} or η_{ki} are distinct; for equal eigenvalues, one should employ other forms (Bellman, 1970).

Setting $\xi = \xi_i$ in Eqs. (27) and (28), we obtain

$$\mathbf{T}_{1ni}(\xi_i) = \exp(\mathbf{M}_{1ni}\xi_i)\mathbf{T}_{1ni}(0) \quad (n = 1, 2, 3, \ldots;\ i = 1, 2, \ldots, p), \tag{33}$$

$$\mathbf{T}_{2ni}(\xi_i) = \exp(\mathbf{M}_{2ni}\xi_i)\mathbf{T}_{2ni}(0) \quad (n = 0, 1, 2, \ldots;\ i = 1, 2, \ldots, p). \tag{34}$$

Thus, we have established the relations between the state variables at the outer surface and those at the inner surface of the ith spherical layer. By utilizing the continuity conditions at each interface, one can finally get the following equations:

$$\mathbf{T}_{1np}(\xi_p) = \mathbf{S}_{1n}\mathbf{T}_{1n1}(0) \quad (n = 1, 2, 3, \ldots), \tag{35}$$

$$\mathbf{T}_{2np}(\xi_p) = \mathbf{S}_{2n}\mathbf{T}_{2n1}(0) \quad (n = 0, 1, 2, \ldots), \tag{36}$$

where $\mathbf{S}_{1n} = \prod_{i=p}^{1}\exp(\mathbf{M}_{1ni}\xi_i)$ and $\mathbf{S}_{2n} = \prod_{i=p}^{1}\exp(\mathbf{M}_{2ni}\xi_i)$ are the square matrices of the second-order and sixth-order, respectively. Through the two matrices, the relations between the boundary state variables at the outer and inner surfaces of a multi-layered piezoelectric hollow sphere are founded. It is noted here that for a specified boundary value problem, one does not need to solve a second-order or (and) a sixth-order linear algebraic equation(s). For example, when the stresses and normal electric displacement are prescribed at both surfaces, i.e. when $T_{kn1}(0)$ and $T_{knp}(\xi_p)$, $(k = 1, 3, 4, 7)$ are known, we can obtain the following equations from Eqs. (35) and (36):

$$S_{1n12}T_{2n1}(0) = T_{1np}(\xi_p) - S_{1n11}T_{1n1}(0) \quad (n = 1, 2, 3, \ldots), \tag{37}$$

$$\begin{bmatrix} S_{2n13} & S_{2n14} & S_{2n16} \\ S_{2n23} & S_{2n24} & S_{2n26} \\ S_{2n53} & S_{2n54} & S_{2n56} \end{bmatrix} \begin{Bmatrix} T_{5n1}(0) \\ T_{6n1}(0) \\ T_{8n1}(0) \end{Bmatrix} = \begin{Bmatrix} T_{3np}(\xi_p) \\ T_{4np}(\xi_p) \\ T_{7np}(\xi_p) \end{Bmatrix} - \begin{bmatrix} S_{2n11} & S_{2n12} & S_{2n15} \\ S_{2n21} & S_{2n22} & S_{2n25} \\ S_{2n51} & S_{2n51} & S_{2n55} \end{bmatrix} \begin{Bmatrix} T_{3n1}(0) \\ T_{4n1}(0) \\ T_{7n1}(0) \end{Bmatrix}$$
$$(n = 1, 2, 3, \ldots), \tag{38}$$

where S_{1nij} and S_{2nij} are elements on the ith row and jth column of the matrices $\mathbf{S}_{1n}$ and $\mathbf{S}_{2n}$, respectively. It is noted that Eq. (38) is not valid for $n=0$ since we have $S_{2n53}=S_{2n54}=S_{2n56}=0$ in this case. In fact, one can readily verify that the matrix $\mathbf{S}_{20}$ has the following form:

$$\mathbf{S}_{20}=\begin{bmatrix} \times & 0 & 0 & \times & \times & 0 \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & 0 & 0 & \times & \times & 0 \\ 0 & 0 & 0 & 0 & \times & 0 \\ \times & 0 & 0 & \times & \times & 1 \end{bmatrix}, \tag{39}$$

where $\times$ represents an arbitrary value. Therefore, when $n=0$, from the first, the fourth, the fifth and the sixth equations in Eq. (36), one obtains

$$\begin{Bmatrix} T_{30p} \\ T_{60p} \\ T_{70p} \\ T_{80p} \end{Bmatrix}=\begin{bmatrix} S_{2011} & S_{2014} & S_{2015} & 0 \\ S_{2041} & S_{2044} & S_{2045} & 0 \\ 0 & 0 & S_{2055} & 0 \\ S_{2061} & S_{2064} & S_{2065} & 1 \end{bmatrix}\begin{Bmatrix} T_{301} \\ T_{601} \\ T_{701} \\ T_{801} \end{Bmatrix}. \tag{40}$$

Eq. (40) can also be derived from the fact that $T_{40}=T_{50}=0$ as one can see from Eqs. (5) and (6) that T_{40} and T_{50} contribute nothing to the elasto-electric field. From the equilibrium condition concerning the electric charge, the third equation in Eq. (40) should be automatically satisfied with $S_{2055}=a_1/b_p$. It is obvious that T_{601} and T_{60p} can be solved from the first and second equations in Eq. (40):

$$T_{601}=(T_{30p}-S_{2011}T_{301}-S_{2015}T_{701})/S_{2014}, \tag{41}$$

$$T_{60p}=S_{2041}T_{301}+S_{2044}T_{601}+S_{2045}T_{701}. \tag{42}$$

From the fourth equation in Eq. (40), one can determine the difference of electric potential

$$T_{80p}-T_{801}=S_{2061}T_{301}+S_{2064}T_{601}+S_{2065}T_{701}. \tag{43}$$

After the state variables of the inner surface are solved, their values at any interior point can be obtained by

$$\mathbf{T}_{1nj}(\xi)=\exp(\mathbf{M}_{1nj}\xi)\prod_{i=j-1}^{1}\exp(\mathbf{M}_{1ni}\xi_i)\mathbf{T}_{1n1}(0) \quad (n=1,2,3,\ldots;\ 0\leqslant\xi\leqslant\xi_j), \tag{44}$$

$$\mathbf{T}_{2nj}(\xi)=\exp(\mathbf{M}_{2nj}\xi)\prod_{i=j-1}^{1}\exp(\mathbf{M}_{2ni}\xi_i)\mathbf{T}_{2n1}(0) \quad (n=0,1,2,\ldots;\ 0\leqslant\xi\leqslant\xi_j). \tag{45}$$

Three induced variables $\Sigma_{\theta\theta}$, $\Sigma_{\phi\phi}$ and $\Sigma_{\theta\phi}$ are determined by

$$\begin{aligned} &\Sigma_{\theta\theta}-\Sigma_{\phi\phi}=2c_{66}\left(\nabla_1^2G-2\frac{\partial^2G}{\partial\theta^2}+2\cot\theta\csc\theta\frac{\partial\psi}{\partial\phi}-2\csc\theta\frac{\partial^2\psi}{\partial\theta\,\partial\phi}\right), \\ &\Sigma_{\theta\theta}+\Sigma_{\phi\phi}=2\beta\Sigma_{rr}+2\gamma\Delta_r+k_1\nabla_1^2G-2k_1w, \\ &\Sigma_{\theta\phi}=-c_{66}\left(\nabla_1^2\psi-2\frac{\partial^2\psi}{\partial\theta^2}-2\cot\theta\csc\theta\frac{\partial G}{\partial\phi}+2\csc\theta\frac{\partial^2G}{\partial\theta\,\partial\phi}\right). \end{aligned} \tag{46}$$

To obtain the other two induced variables Δ_θ and Δ_ϕ in terms of the state variables, we first employ the following separation formula:

$$\Delta_\theta = -\frac{1}{\sin\theta}\frac{\partial \Delta_1}{\partial \phi} - \frac{\partial \Delta_2}{\partial \theta}, \qquad \Delta_\phi = \frac{\partial \Delta_1}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial \Delta_2}{\partial \phi}. \tag{47}$$

Then, we have

$$\Delta_1 = (e_{15}/c_{44})\Sigma_1, \qquad \Delta_2 = (e_{15}/c_{44})\Sigma_2 + k_3\Phi. \tag{48}$$

6. Numerical examples

Three piezoelectric materials, PZT-4, PZT-7A and $BaTiO_3$, will be considered in the following. The material constants of these materials can be found in Dunn and Taya (1994) and are given in Table 1 here for the readers' convenience. Numerical calculation is first carried out for checking the piezoelectric effect on the stresses and displacements of a homogeneous hollow sphere under a uniform external pressure q. The inner radius of the sphere, a, is a half of the outer radius b, i.e. $a = 0.5b$.

Figs. 2 and 3 show the distributions of the non-dimensional normal stresses σ_{rr}/q and τ/q ($\tau = \sigma_{\theta\theta} = \sigma_{\phi\phi}$) in a PZT-4 hollow sphere as well as in the corresponding elastic sphere (the piezoelectric effect in PZT-4 is neglected) which is denoted as PZT-4(E) in both figures. It is noted here that Saint-Venant has obtained an exact closed-form solution to the problem of a spherically isotropic elastic hollow sphere subjected to both uniform inner and outer pressures (Love, 1927; Lekhnitskii, 1981). Heyliger and Wu (1999) also presented an analytic solution to the spherically axisymmetric problem of a piezoelectric sphere. Our results for the PZT-4(E) sphere and for the PZT-4 piezoelectric sphere are found identical to the Saint-Venant's solution and the solution of Heyliger and Wu (1999), respectively. It is seen from Fig. 2 that the piezoelectric effect lead to a small increase of the normal compression stress ($-\sigma_{rr}$) in the sphere. Fig. 3 shows that the piezoelectric effect makes the difference between the other two normal stresses at the inner and outer spherical surfaces of the PZT-4 sphere greater than that of the PZT-4(E) sphere. Calculation has also been made for the other two piezoelectric materials PZT-7A and $BaTiO_3$ with similar observations obtained. Since the corresponding curves are very close if they are put in figures simultaneously, they are not presented here. However, for the non-dimensional normal displacement $\bar{u}_r = wc_{44}^{(1)}/(bq)$, the curves can be clearly shown in one figure as we can see from Fig. 4. We find that the piezoelectric effect can improve the anti-deformation ability of the sphere, especially for PZT-4 material.

For the multilayered case, a three-layered hollow sphere with the following geometry was considered,

$$a_1 = a = 0.5b, \quad a_2 = b_1 = 0.7b, \quad a_3 = b_2 = 0.8b, \quad b_3 = b.$$

Three different cases of material combination of the three layers are considered: (A) $BaTiO_3$/$BaTiO_3$/$BaTiO_3$, i.e. the sphere is homogeneous; (B) $BaTiO_3$/PZT-4/$BaTiO_3$; (C) $BaTiO_3$/PZT-7A/$BaTiO_3$.

Table 1
Material constants

Materials	PZT-4	PZT-7A	$BaTiO_3$
c_{ij} (10^{10} N m^{-2})	$c_{11} = 13.9$, $c_{12} = 7.78$, $c_{13} = 7.43$, $c_{33} = 11.5$, $c_{44} = 2.56$	$c_{11} = 14.8$, $c_{12} = 7.62$, $c_{13} = 7.42$, $c_{33} = 13.1$, $c_{44} = 2.54$	$c_{11} = 15.0$, $c_{12} = 6.6$, $c_{13} = 6.6$, $c_{33} = 14.6$, $c_{44} = 4.4$
e_{ij} (C m^{-2})	$e_{15} = 12.7$, $e_{31} = -5.2$, $e_{33} = 15.1$	$e_{15} = 9.7$, $e_{31} = -2.1$, $e_{33} = 9.5$	$e_{15} = -0.21$, $e_{31} = -0.24$, $e_{33} = 0.44$
ε_{ij} (10^{-10} F m^{-1})	$\varepsilon_{11} = 64.64$, $\varepsilon_{33} = 56.22$	$\varepsilon_{11} = 40.73$, $\varepsilon_{33} = 20.81$	$\varepsilon_{11} = 98.72$, $\varepsilon_{33} = 111.56$

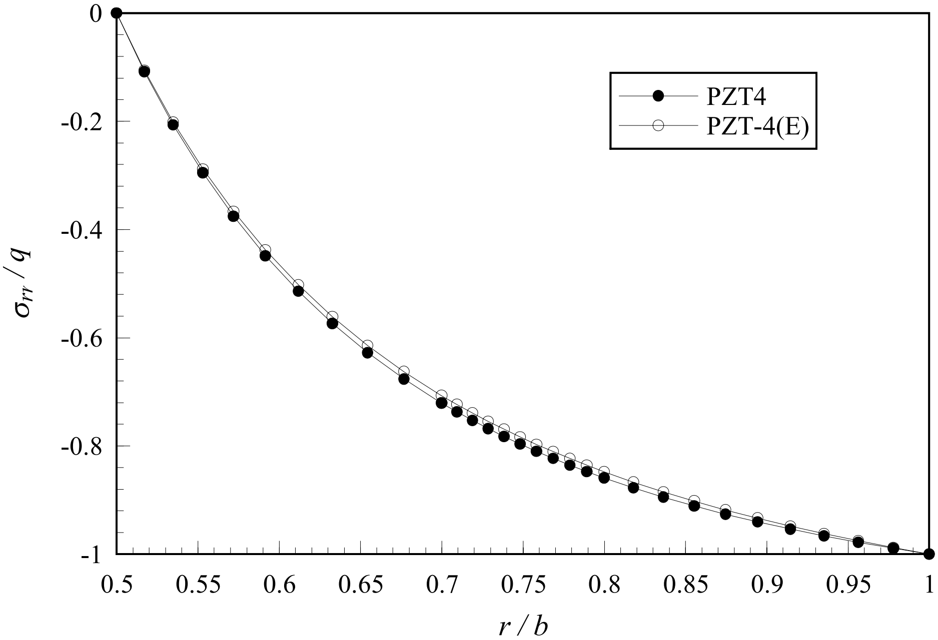

Fig. 2. Piezoelectric effect on σ_{rr}/q in a homogeneous hollow sphere under uniform external pressure.

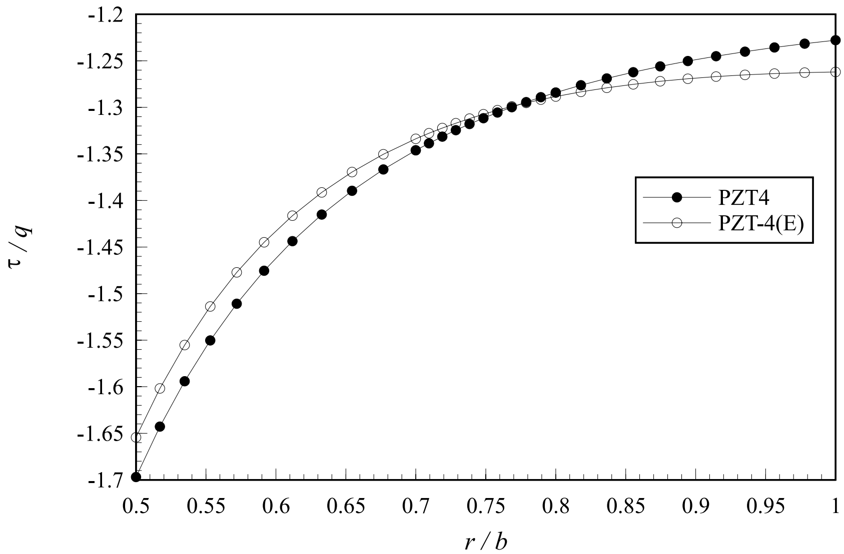

Fig. 3. Piezoelectric effect on τ/q in a homogeneous hollow sphere under uniform external pressure.

It is assumed that the sphere is subjected to a distributed uniform pressure q over the ranges $0 \leqslant \theta \leqslant \theta_0$ and $\pi - \theta_0 \leqslant \theta \leqslant \pi$ at the outer surface $r = b$ (Fig. 5). From Fig. 5, one has $a = a_1$, $b = b_3$, and $h = b/k = (1 - \cos\theta_0)b$. Obviously when $k = 1$, the whole outer surface will undergo a uniform pressure,

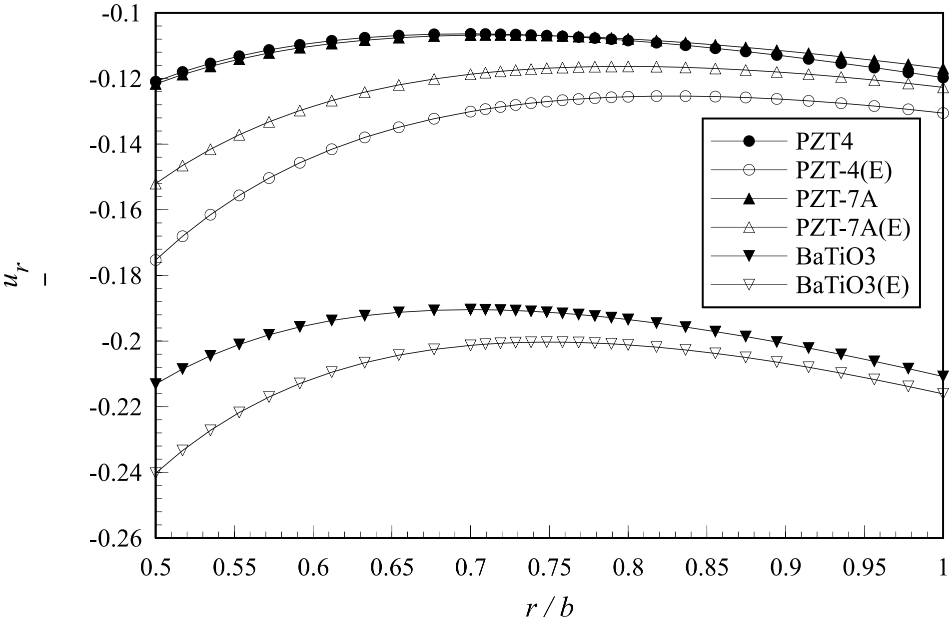

Fig. 4. Piezoelectric effect on $\bar{u}_r$ in a homogeneous hollow sphere under uniform external pressure.

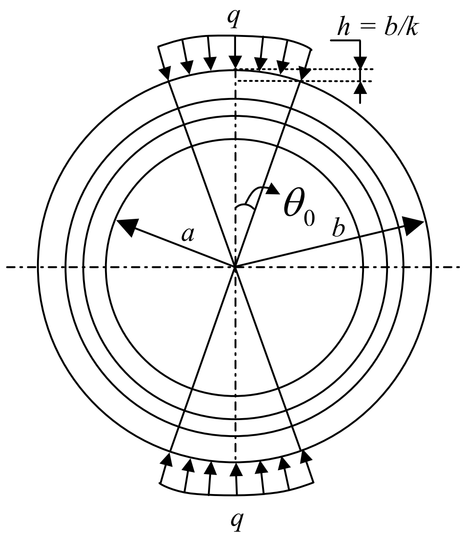

Fig. 5. A three-layered hollow sphere under distributed pressure.

while for $k \to \infty$, the sphere will be subjected to a couple of concentrated forces applied at the two poles. In the following, we will take the value $k = 4$. Because the problem considered is axisymmetric for which one has $m = 0$ in Eq. (24), the distributed pressure can be expanded in terms of Legendre polynomials as $\sum_{n=0}^{\infty} \alpha_n P_n(\cos\theta)$, where the coefficients α_n are given by

$$\alpha_n = \begin{cases} q/k, & n = 0 \\ \left[1-(-1)^{n+1}\right]\left[P_{n-1}\left(\frac{k-1}{k}\right) - P_{n+1}\left(\frac{k-1}{k}\right)\right]\frac{q}{2}, & n > 0 \end{cases} \tag{49}$$

Figs. 6–8 give the distribution curves of the non-dimensional stresses σ_{rr}/q, $\sigma_{\theta\theta}/q$ and $\sigma_{\phi\phi}/q$, respectively, along the radial direction when $\theta = \pi/6$. Figs. 7 and 8 show that both $\sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$ have a sudden jump across the material interface for Cases B and C. Though for all three cases, the distributions of stresses

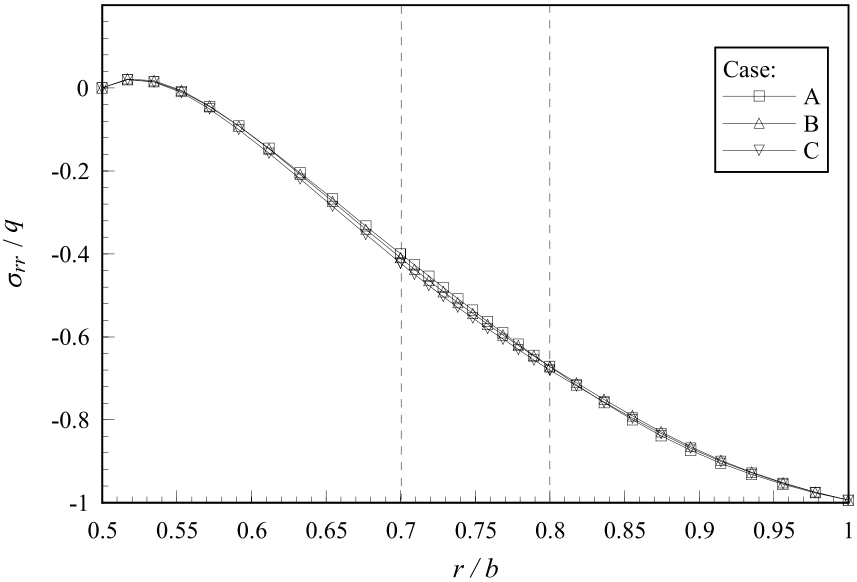

Fig. 6. Distribution of σ_{rr}/q in the radial direction ($k = 4$, $\theta = \pi/6$).

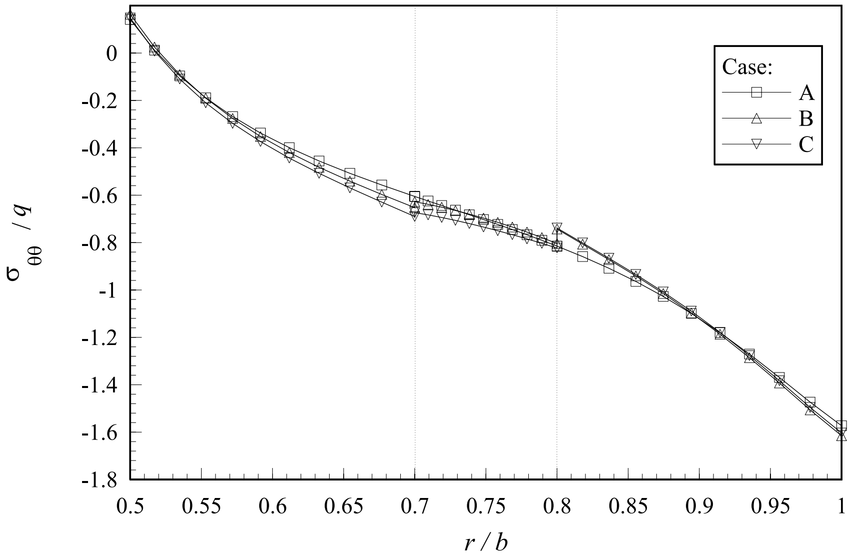

Fig. 7. Distribution of $\sigma_{\theta\theta}/q$ in the radial direction ($k = 4$, $\theta = \pi/6$).

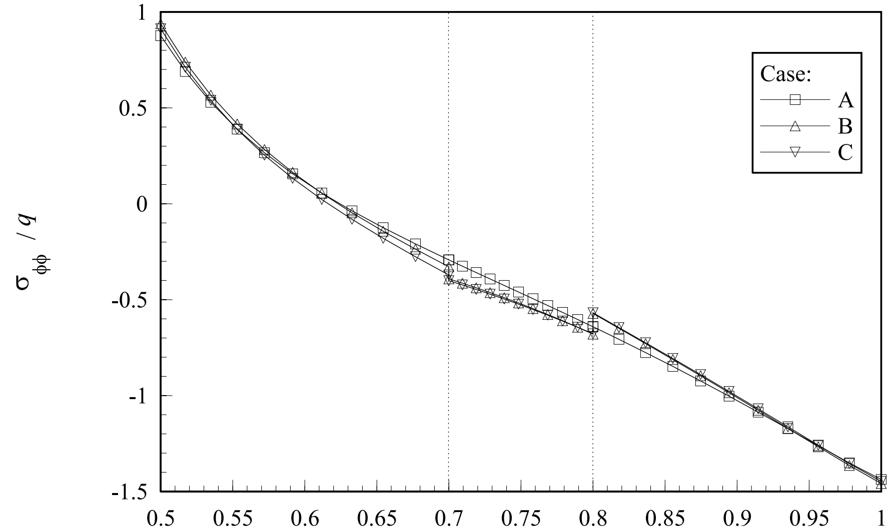

Fig. 8. Distribution of $\sigma_{\phi\phi}/q$ in the radial direction ($k = 4,\ \theta = \pi/6$).

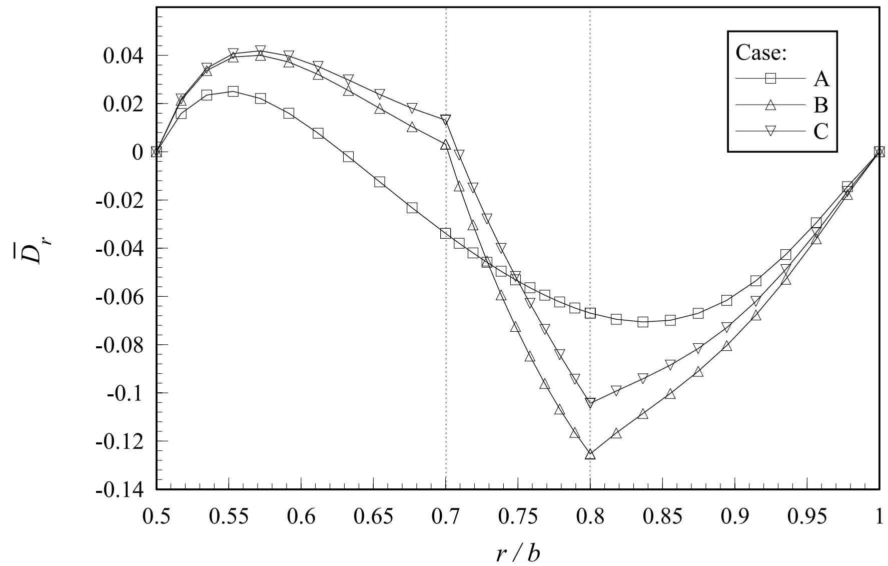

Fig. 9. Distribution of $\overline{D}_r$ in the radial direction ($k = 4,\ \theta = \pi/6$).

differ from each other slightly, the material combination does have a significant effect on the distributions of the mechanical displacement and the electric displacement. Figs. 9 and 10 show the distribution curves of the non-dimensional electric displacement $\overline{D}_r = D_r c_{44}^{(1)}/[qe_{33}^{(1)}]$ and the non-dimensional mechanical displacement $\bar{u}_r = wc_{44}^{(1)}/(bq)$, respectively, along the radial direction when $\theta = \pi/6$. The differences between the three cases are clearly shown in both figures. The circumferential distributions of σ_{rr}/q and $\overline{D}_r = D_r c_{44}^{(1)}/[qe_{33}^{(1)}]$ at the interface $r = 0.7b$ are shown in Figs. 11 and 12, respectively. The difference of the circumferential distribution of $\overline{D}_r$ between the three cases is also much obvious than that of σ_{rr}/q.

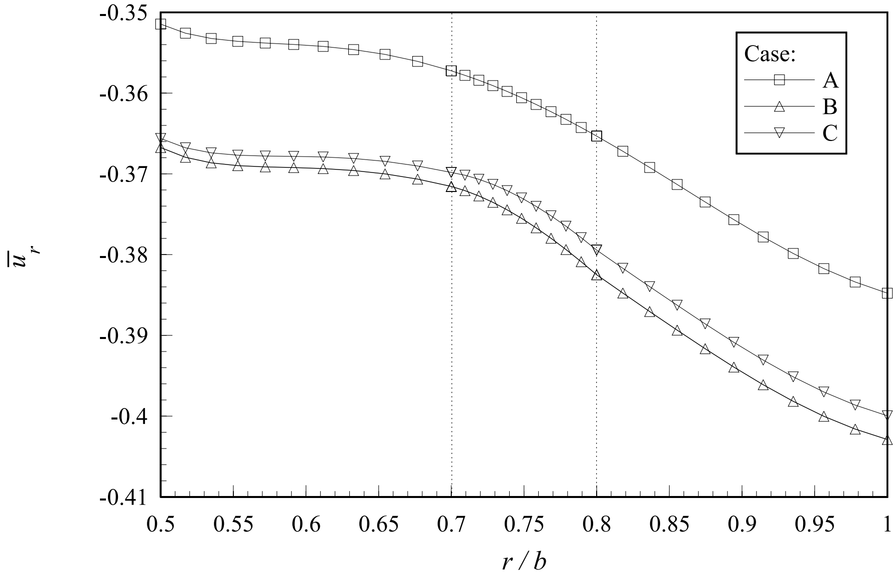

Fig. 10. Distribution of $\bar{u}_r$ in the radial direction ($k = 4,\ \theta = \pi/6$).

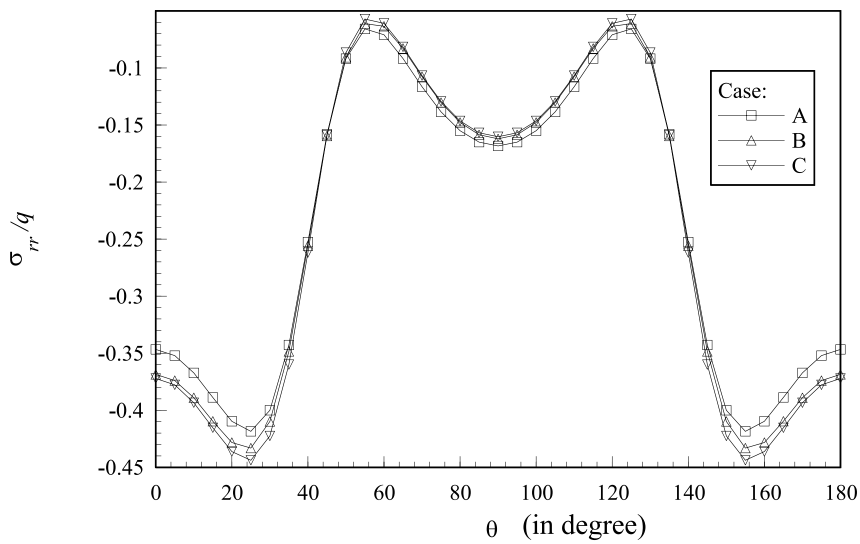

Fig. 11. Distribution of σ_{rr}/q in the circumferential direction ($k = 4,\ r = 0.7b$).

7. Conclusion

This paper develops a state space method to exactly analyze the statics of multi-layered piezoelectric hollow spheres. It is shown that two separated state equations with constant variables can be derived using a series of techniques. Relations between the state variables at the inner and outer spherical boundary

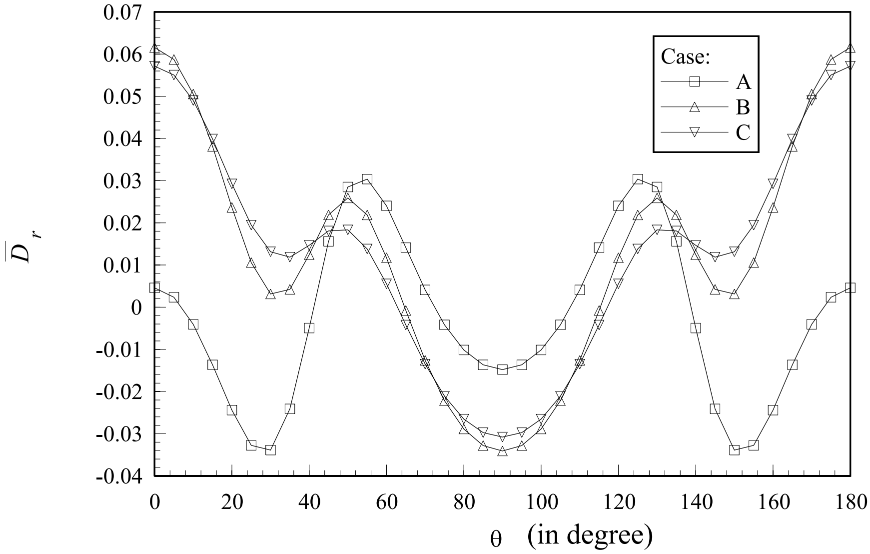

Fig. 12. Distribution of $\overline{D}_r$ in the circumferential direction ($k = 4,\ r = 0.7b$).

surfaces are established in the paper, from which any boundary value problem of a multi-layered hollow sphere can be readily dealt with. Numerical examples are given in the paper to show the effectiveness of the method. In particular, comparisons with those of the Saint-Venant's solution for an elastic sphere and with those of the exact solution obtained in Heyliger and Wu (1999) for a piezoelectric sphere are made and excellent agreement is obtained.

Since the present method is completely based on the three-dimensional exact equations for linear piezoelasticity, it can be a benchmark for assessing any two-dimensional approximate shell theory or numerical method.

Acknowledgements

The work was supported by the National Natural Science Foundation of China (No. 19872060).

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